time Δv_i is changed, the elastic constants are completely redetermined, and this causes a corresponding change in the adjusted values $v_{i,\text{calc}}$. The errors of the six elastic constants Δc_{ij} were reduced by the misorientation correction for bismuth.

In the case of antimony χ^2 became slightly worse by 7% (1.58 to 1.71) and the errors of the elastic stiffness constants for c_{11}, c_{13}, and c_{33} increased slightly, while those for c_{14}, c_{44}, and c_{66} decreased slightly. The trace relations for both semimetals were improved slightly by the application of this correction.

Papadakis has calculated curves for correcting v_7 for diffraction.[6] His graph is plotted in terms of the phase shift between echoes as a function of $S=z\lambda/a^2$, where z= path length in sample, λ= wavelength, and a= piston (transducer) radius for Waterman's anisotropy parameter b (the coefficient of $\theta^2 v_7$ in Appendix A). The calculation was not poslible as our b value for antimony is -9.8, which exceeds Papadakis's maximum value of -5. However, assuming the latter anisotropy parameter, the velocity correction for v_7 amounts to -0.1%.

CALCULATIONS AND RESULTS

The fourteen equations of Eckstein, Lawson, and Reneker (ELR)[2] were used by one of us (ERC) to determine the six elastic constants, by the method of least squares.[5] A preliminary calculation indicated that the measured velocity v_{11} (see Table I) was inconsistent, a conclusion which was apparent from the trace relations. It was therefore omitted and the remaining thirteen velocities were used to compute a "best" set of elastic constants: $c_{11}=101.3\pm1.6$; $c_{13}=29.2\pm2.2$; $c_{33}=45.0\pm1.5$; $c_{44}=39.3\pm0.7$; $c_{14}=20.9\pm0.4$; $c_{66}=33.4\pm0.6$; and $c_{12}=34.5\pm2.0$ all in units of 10^{10} dyn cm^{-2}. The isothermal correction is negligible. The importance of using a least-squares adjustment of the data in order to determine the elastic constants is that the adjusted velocities which may then be evaluated will satisfy *exactly* all the trace relations in theory.

The trace for the principal-axis-cut crystal of antimony is $T_{xy}=(26.00\pm0.24)\times10^{10}$ cm^2/sec^2 using the adjusted values from Table I, and the diagonal trace for the 45°-cut crystal is $T_{45°}=(22.22\pm0.19)\times10^{10}$ cm^2/sec^2.

Table I gives the values of the fourteen measured velocities and their least-squares adjusted values. The errors assigned to the adjusted velocities are computed from the full error matrix of the least-squares adjustment and should be used with care since the data are inter-related and cannot be treated as statistically independent. The elastic stiffness constant c_{13} as a result of this computation is found to be 10% higher than the previous finding.[1]

The compliances are $s_{11}=16.31$; $s_{33}=30.96$; $s_{44}=38.14$; $s_{12}=-6.15$; $s_{66}=44.93$; $s_{13}=-6.60$; and $s_{14}=-11.95$ all in units of 10^{-13} cm^2/dyn, and are in fair agreement with the data of Bridgman[7] who obtains $s_{11}=17.7$; $s_{33}=33.8$; $s_{44}=41$; $s_{12}=-3.8$; $s_{66}=43$; $s_{13}=-8.5$; and $s_{14}=-8.0$, all in units of 10^{-13} cm^2/dyn.

The fourteen equations were used with the same least-squares procedure as above to determine the six elastic constants of bismuth. The data of Eckstein, Lawson, and Reneker determined by the pulse-echo technique were used.[2] They state their velocities are accurate to better than 1%, and their principal error arises from the transducer transit-time correction. The "best" set of elastic constants is $c_{11}=63.7\pm0.2$; $c_{13}=24.7\pm0.2$; $c_{33}=38.2\pm0.2$; $c_{44}=11.23\pm0.04$; $c_{14}=7.17\pm0.04$; $c_{66}=19.41\pm0.06$; $c_{12}=24.9\pm0.2$, all in units of 10^{10} dyn/cm^2.

The compliances are $s_{11}=25.7$; $s_{33}=40.83$; $s_{44}=116.4$; $s_{12}=-8.13$; $s_{66}=67.6$; $s_{14}=-21.6$; $s_{13}=-11.33$, all in units of 10^{-13} cm^2/dyn. Bridgman's[8] results are $s_{11}=26.9$; $s_{33}=28.7$; $s_{44}=104.8$; $s_{12}=-14.0$; $s_{66}=81.2$; $s_{14}=16.0$; $s_{13}=-6.2$, all in units of 10^{-13} cm^2/dyn.

The principal-axis trace relation, or T_{xy} gives $(9.623\pm0.041)\times10^{10}$ cm^2/sec^2 from our least-squares results, versus $T_x=9.580$ and $T_y=9.654\times10^{10}$ cm^2/sec

TABLE II. Elastic-stiffness constants c_{ij} (10^{10} dyn/cm^2) of antimony at room temperature.

c_{11}	c_{13}	c_{14}	c_{33}	c_{44}	c_{66}	χ^2	Remarks
99.4(1)	26.4(4)	21.6(4)	44.5(9)	39.5(5)	34.2(5)	6.4	Near least squares.[a] No correction to experimental data.[b]
99.5±2.2	25.3±2.6	21.5±0.6	45.0±0.9	40.3±0.7	33.9±0.7	4.9	Least squares (ERC). No correction to experimental data.[b]
101.4±1.5	29.4±2.1	20.9±0.5	45.0±1.4	39.2±0.8	33.4±0.8	1.58	Least squares (ERC). Data corrected for "transit time."[c,d]
101.3±1.6	29.2±2.2	20.9±0.4	45.0±1.5	39.3±0.7	33.4±0.6	1.71[e]	Same as preceding plus an added misorientation correction.[f]

[a] Described in Ref. 1.
[b] Made by the rf pulse-echo technique (longitudinal principally at 10 MHz; shear principally at 5 MHz) of Ref. 1.
[c] Arbitrarily assumed to be ±1 cycle of pulse. See Ref. 4.
[d] The basic experimental data for this paper is slightly different from that chosen in Ref. 1.
[e] Data chosen as best in this paper.
[f] Equations (A5)–(A10).

[6] Emmanuel P. Papadakis (private communication).
[7] P. W. Bridgman, Proc. Am. Acad. Arts Sci. 60, 363 (1925).
[8] Reference 7, p. 305.

TABLE III. Elastic constants c_{ij}(10^{10} dyn/cm²) of bismuth at room temperature (experimental measurements of ELR used[a]).

c_{11}	c_{13}	c_{14}	c_{33}	c_{44}	c_{66}	χ^2	Remarks
63.50	24.50	7.23	38.10	11.30	19.40	1.7	No least squares. Transducer transit-time correction.[a]
63.22	24.40	7.20	38.11	11.30	19.40	2.6	Near least squares.[b] Transducer transit-time correction.[a]
63.7±0.3	24.6±0.2	7.20±0.04	38.1±0.2	11.26±0.04	19.38±0.07	1.4	Near least squares (ERC), plus preceding correction.
63.7±0.2	24.7±0.2	7.17±0.04	38.2±0.2	11.23±0.04	19.41±0.06	0.93[c]	Same as preceding plus misorientation correction.[d]

[a] See Ref. 2. Ultrasonic video pulse-echo technique used at 12 MHz.
[b] Described in Ref. 1.
[c] Data chosen as best in this paper.
[d] Equations (A5)–(A10).

for Eckstein, Lawson, and Reneker.[2] The 45°-cut crystal trace relation gives $T_{45°}=(7.911\pm0.022)\times10^{10}$ cm²/sec² from least squares, while ELR obtain $T_{45°}=7.974$ for $\varphi=90°$, and $T_{135°}=7.899\times10^{10}$ cm²/sec² for $\varphi=-90°$.

Tables II and III summarize the effects of different data processing on the elastic-stiffness constants of antimony and bismuth, respectively, and represent additional measurements on the original specimens taken by one of us (deB).

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APPENDIX A: MISORIENTATION CORRECTION

The velocity error due to misorientation can be calculated from the following determinant for point group $\bar{3}m$, where $x=\rho v^2$, ρ is the density, and v the sonic velocity.[9]

$$\begin{vmatrix} A-x & H & G \\ H & B-x & F \\ G & F & C-x \end{vmatrix} = 0\,. \tag{A1}$$

Here,

$$A=c_{11}\sin^2\theta\cos^2\varphi+\tfrac{1}{2}(c_{11}-c_{12})\sin^2\theta\sin^2\varphi+c_{44}\cos^2\theta+2c_{14}\sin\theta\cos\theta\sin\varphi\,,$$

$$B=\tfrac{1}{2}(c_{11}-c_{12})\sin^2\theta\cos^2\varphi+c_{11}\sin^2\theta\sin^2\varphi+c_{44}\cos^2\theta-2c_{14}\sin\theta\cos\theta\sin\varphi\,,$$

$$C=c_{44}\sin^2\theta+c_{33}\cos^2\theta\,,$$

$$F=c_{14}\sin^2\theta(1-2\sin^2\varphi)+(c_{13}+c_{44})\sin\theta\cos\theta\sin\varphi\,,$$

$$G=2c_{14}\sin^2\theta\sin\varphi\cos\varphi+(c_{13}+c_{44})\sin\theta\cos\theta\cos\varphi\,,$$

and

$$H=\tfrac{1}{2}(c_{11}+c_{12})\sin^2\theta\sin\varphi\cos\varphi+2c_{14}\sin\theta\cos\theta\cos\varphi\,.$$

θ is the angle between a direction of propagation and the positive z axis, φ is the angle in the basal plane measured from the positive x axis counterclockwise to the projection of the direction of propagation on the basal plane.

The equations for the three velocities are, neglecting G and H which are small and would be zero for a perfectly oriented principal axis and 45°-cut crystal,

$$\rho v^2=A\,, \tag{A2}$$

$$\rho v^2=(B+C)+\{(B-C)^2+4F^2\}^{1/2}\,, \tag{A3}$$

$$\rho v^2=(B+C)-\{(B-C)^2+4F^2\}^{1/2}\,. \tag{A4}$$

Equations (A2)–(A4) were differentiated with respect to velocity in terms of θ and φ, and solved for the error $\pm\Delta v_i$ by inserting the appropriate elastic-stiffness constants c_{ij}, velocities v_i, and the value $\Delta\theta=\pm1°$ for

TABLE IV. Velocity errors due to misorientation for antimony and bismuth.

Symbol	Antimony 10^5 cm/sec	Bismuth 10^5 cm/sec
$\pm\Delta v_1$	0.00	0.000
$\pm\Delta v_2$	0.00	0.000
$\pm\Delta v_3$	0.00	0.000
$\pm\Delta v_4$	0.00	0.000
$\pm\Delta v_5$	0.00	0.000
$\pm\Delta v_6$	0.00	0.000
$\pm\Delta v_7$	0.00	0.000
$\pm\Delta v_8$	0.02	0.012
$\pm\Delta v_9$	0.01	0.006
$\pm\Delta v_{10}$	0.00	0.005
$\pm\Delta v_{11}$	0.03	0.010
$\pm\Delta v_{12}$	0.01	0.008
$\pm\Delta v_{13}$	0.01	0.008
$\pm\Delta v_{14}$	0.03	0.002

[9] J. B. Wachtman, Jr., W. E. Tefft, D. G. Lam, Jr., and R. P. Stinchfield, J. Res. Natl. Bur. Std. 64A, 219 (1960).